DYNAMICS OF A CHARGED GAS CLOUD IN A MAGNETIC FIELD

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There are several approaches to the expansion of a gas cloud, of which the best with regard to derivation of analytic solutions is that based on [1], where an initial general system of equations was derived for the elements of the matrix that defines the solution to the equations of hydrodynamics, and it was pointed out that several first integrals of this system can be derived. With regard to the escape of the gas cloud into vacuum, several first integrals have been derived [2, 3], and the system has been integrated completely in certain simple cases. In [4], allowance was made for the gravitational interaction in this model, and there was a detailed discussion of the behavior of the solution on the basis of the qualitative theory of differential equations. It is also possible to integrate the system in certain cases by time transformation. In [5] the solution to the system of [1] was applied to the analysis of some features of a plasma cloud in a laser thermonuclear synthesis. Therefore, the approach of [1] provided some interesting results in plasma physics and astrophysics. The scope for using this approach in these areas is far from exhausted, since the exact solutions correspond to particular situations, where the system of equations simplifies very considerably. Another sphere of application is to the motion of a cloud of ideal gas containing charged particles in an external steady magnetic field. This form of problem arises in astrophysics and in plasma physics, where a solution may assist in qualitative explanation of the behavior of objects as in controlled thermonuclear synthesis. It is possible to extend the application in this way because the additional term produced by the magnetic field in the equation of motion is proportional to the coordinate of the particle (Lagrangian coordinate), which is a necessary condition for solving the system by the method of [1].

Here we consider the dynamics of a cloud of charged particles taking the form of an ellipsoid in an external magnetic field whose direction coincides with one of the axes of the ellipsoid (with the longitudinal axis for a spheroid).

The axis of rotation of the cloud coincides in direction with the magnetic field. The gas is considered ideal and the expansion is adiabatic. We convert from the cartesian coordinates x_i to the Lagrangian ones a_i for particular parts of the gas:

$$x_i(t) = F_{ik}(t)a_k \tag{1}$$

(here and subsequently the repeating subscripts indicate summation from 1 to 3) and we get [2] what follows from the equation continuity, Poisson's equation, and the first law of thermodynamics:

$$\rho = \rho_0(a)\varphi_0/\varphi,\ p = (\gamma - 1)\rho_0(a)U_0(a)(\varphi_0/\varphi)^\gamma,$$
$$\varphi = \det||F_{ik}||,\ v_i = \dot{F}_{ik}a_k,$$

where ρ is gas density; γ, adiabatic parameter; U, specific internal energy of the gas; and v_i, velocity of a part. Quantities with subscript 0 correspond to initial values. Equation (1) is applicable to a magnetic field of this configuration, because the parts of the gas rotate synchronously in orbits in the magnetic field, and the velocities are proportional to the radii, which means that (1) is satisfied. The momentum equation in the presence of the field is as follows on the basis of (1):

$$\rho\ddot{F}_{ik}a_k = -F_{ki}^{-1}\frac{\partial}{\partial a_k}\left[(\gamma-1)\rho_0(a)U_0(a)\left(\frac{\varphi_0}{\varphi}\right)^\gamma\right] + \rho\frac{e}{mc}\varepsilon_{ihl}B_l\dot{F}_{hm}a_m,$$

where B is the induction; ε_{ikl}, a completely antisymmetric unit pseudotensor of the third rank; e, particle charge; and m, particle mass. The variables can be separated into two cases: when $\rho_0(a)$ takes the form

$$\rho_0(a) = \rho_{00}\exp\{-a^2/a_0^2\}$$

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(it is assumed for simplicity that the level surfaces are spherical in Lagrangian coordinates), and $U_0(a)$, when

$$\rho_0(a) = \text{const}, \quad U_0(a) = U_{00}\left(1 - a^2/a_0^2\right) \text{(here } a^2 = a_1^2 + a_2^2 + a_3^2\text{)}.$$

These two cases correspond to different density and temperature distributions in space. We consider one of the cases (for definiteness the second, but we recall that the formulas for the two cases are practically the same), which gives us a system of equations of the type of [1]:

$$\ddot{F}_{ik} = \alpha \frac{F_{ki}^{-1}}{\varphi^{\gamma-1}} + \varepsilon_{ihl} B_l \dot{F}_{mk} \frac{e}{mc}, \quad \alpha = 2(\gamma - 1) U_{00} \varphi_0^{\gamma-1}/a_0^2. \tag{2}$$

System (2) has several first integrals. We transform by analogy with [2] to get the energy-conservation equation:

$$\dot{F}_{ik}^2 = -\frac{2}{\gamma - 1}\frac{\alpha}{\varphi^{\gamma-1}} + C_1. \tag{3}$$

Integrals analogous to the laws of conservation of momentum and vorticity [2] take the form

$$\dot{F}_{ik} F_{il} - F_{ik}\dot{F}_{il} = \frac{e}{mc} B_n \varepsilon_{imn} F_{mk} F_{il} + C_2, \quad \frac{e}{mc}\varepsilon_{lim} B_m (\dot{F}_{ik} F_{lk} - F_{ik}\dot{F}_{lk}) = -\left(\frac{e}{mc}\right)^2 \varepsilon_{iln}\varepsilon_{ipq} B_n B_q F_{pk} F_{lk} + 2C_3. \tag{4}$$

The vorticity is not conserved in the form given in [2] here, but the relation of (4) is obeyed. There is also an integral of motion analogous to that found in [3]. In fact, it can be shown that the following applies on the basis of the second equation in (4) when the magnetic field lies along the axis of rotation of the ellipsoid:

$$\frac{1}{2}\left(F_{ik}^2\right)^{\cdot\cdot} = -\frac{1}{2}\left(\frac{eB}{mc}\right)^2 \left(F_{xk}^2 + F_{yk}^2\right) + C_1 + C_3. \tag{5}$$

Here C_1 and C_3 are constants from (3) and (4). In the case of an arbitrary magnetic field, the result is

$$\left(F_{ik}^2\right)^{\cdot\cdot} = -\left(\frac{e}{mc}\right)^2 \varepsilon_{iln}\varepsilon_{ipq} B_n B_q F_{pk} F_{lk} + 2(C_1 + C_3),$$

which shows that it is impossible to obtain an integral of the type of [3]. The integration in (5) can be performed in a fairly general case, which corresponds to specification of the time dependence of F_{zz} in the absence of nondiagonal elements F_{xz}, F_{yz}, F_{zx}, F_{zy} in the F_{ik} matrix. An example of this relationship is provided by

$$\left(F_{zz}^2\right)^{\cdot\cdot} = C_8 \cos(\omega t) + 2C_5. \text{ Here } F_{zz}^2 = C_5 t^2 + C_6 t + C_7 - \frac{C_8}{\omega^2}\cos(\omega t)$$

and we get from (5) for $u = F_{xk}^2 + F_{yk}^2$ that

$$\ddot{u} + \omega_0^2 u = 2(C_1 + C_3 - C_5) - C_8 \cos(\omega, t), \tag{6}$$

where $\omega_0^2 = eB/mc$; the solution to (6) is familiar (the equation for forced oscillations) and for the case of resonance ($\omega = \omega_0$) we get

$$u = \frac{2}{\omega_0^2}(C_1 + C_3 - C_5) + \left(C_{10} - \frac{C_8}{2\omega_0} t\right)\sin(\omega_0 t) + C_{11}\cos(\omega_0 t), \tag{7}$$

which gives the corresponding integral. Note that the case $C_8 = C_5 = C_6 = 0$ corresponds to two-dimensional expansion of an unbounded gas cylinder, while the case in which these quantities are not zero corresponds to expansion of a gas ellipsoid in a longitudinal magnetic field.

The system of equations for F_{ik} can be solved completely in the simplest cases. One of these is a rotating unbounded gas cylinder of circular cross section, where γ is taken as two to make the equations integrable. There are nonzero matrix elements $F_{xx} = F_{yy}$, $F_{xy} = -F_{yx}$. We transform the equations and get from (3) and (7) with $F_{xx} = R \sin\chi$, $F_{xy} = R\cos\chi$ that

$$R = \sqrt{u/2}, \quad \dot{\chi}^2 = \frac{1}{u}\left(C_1 - \frac{2\alpha}{\varphi} - \frac{\dot{u}^2}{4u}\right), \quad \varphi = \frac{u}{2}. \tag{8}$$

We integrate (8) with the initial conditions

$$F_{xx}(0)=F_{yy}(0)=1,\quad \dot F_{xx}(0)=\dot F_{yy}(0)=b,\quad F_{xy}(0)=F_{yx}(0)=\dot F_{xy}(0)=\dot F_{yx}(0)=0,\quad C_1=2(b^2+\alpha),$$

$$C_3=\omega_0^2,\quad \psi=-\arctan\left(\frac{b^2+\alpha}{b\omega_0}\right),\quad u=\frac{2}{\omega_0^2}\left[\omega_0^2+2(b^2+\alpha)\right]+\frac{4\sqrt{b^2\omega_0^2+(b^2+\alpha)^2}}{\omega_0^2}\sin(\omega_0 t+\psi),$$

to get

$$\chi=\frac{\omega_0 t}{2}-\frac{\omega_0}{\sqrt{\omega_0^2+4\alpha}}\arctan\left\{\frac{\left[2(b^2+\alpha)+\omega_0^2\right]\tan\left(\frac{\omega_0 t}{2}+\psi\right)-2\sqrt{b^2\omega_0^2+(b^2+\alpha)^2}}{\omega_0\sqrt{\omega_0^2+4\alpha}}\right\}.$$

The most interesting characteristics of this case, viz., the radius of a cylinder and the angular momentum about the longitudinal axis, are defined by

$$R=\left\{\frac{\omega_0^2+2(b^2+\alpha)}{\omega_0^2}+\frac{2}{\omega_0^2}\sqrt{b^2\omega_0^2+(b^2+\alpha)^2}\sin(\omega_0 t+\psi)\right\}^{1/2},$$

$$M_z=\frac{8\pi}{15}\rho_{00}a_0^5(F_{xx}\dot F_{yx}+F_{xy}\dot F_{yy})=\frac{4\pi}{15}\rho_{00}a_0^5\sqrt{u\left(C_1-\frac{2\alpha}{\psi}-\frac{\dot u^2}{4u}\right)}.$$

Here R represents oscillations about some equilibrium value, and the behavior of M_z is also oscillatory. When $\gamma = 5/3$, the system cannot be integrated to completion in a simple fashion, but R and M_z are readily determined. Here

$$R=\sqrt{u/2},\quad M_z=\frac{4\pi}{15}\rho_{00}a_0^5\left[u\left(C_1-\frac{2\alpha}{\gamma-1}\left(\frac{2}{F_{zz}u}\right)^{\gamma-1}-(\dot F_{zz})^2-\frac{\dot u^2}{4u}\right)\right]^{1/2}.$$

The behavior of these quantities is similar to that given above, but the oscillations are nonstationary, as (7) shows.

The solution for $F_{zz}^2 = u/2$ is also expressible in terms of elementary functions. In that case the equation for χ on rotation of a spheroid around the longitudinal axis simplifies to

$$\dot\chi=\sqrt{\frac{1}{u}\left(C_1-\frac{6\alpha}{u}-\frac{3\dot u^2}{8u}\right)}. \tag{9}$$

The equation for u becomes

$$\frac{3}{2}\ddot u+\omega_0^2 u=2(C_1+C_3). \tag{10}$$

We solve (10) to get

$$u=A\sin\left(\sqrt{\frac{2}{3}}\omega_0 t+\psi\right)+\frac{2}{\omega_0^2}(C_1+C_3). \tag{11}$$

We substitute (11) into (9) and integrate with the initial data

$$F_{xx}(0)=F_{yy}(0)=F_{zz}(0)=1,\quad F_{xy}(0)=F_{yx}(0)=0,\ \dot F_{xy}(0)=\dot F_{yx}(0)=0,$$

$$\dot F_{xx}(0)=\dot F_{yy}(0)=\dot F_{zz}(0)=b,\quad C_1=3(b^2+\alpha),\quad C_3=\omega_0^2,\quad \tan\psi=\sqrt{\frac{3}{2}}\frac{b^2+\alpha}{b\omega_0},$$

$$A=2\sqrt{3}\sqrt{2b^2\omega_0^2+3(b^2+\alpha)^2}/\omega_0^2,$$

to get finally

$$\chi=\frac{\omega_0 t}{2}-\frac{\omega_0}{\sqrt{\omega_0^2+6\alpha}}\arctan\left\{\frac{\left[\omega_0^2+3(b^2+\alpha)\right]\tan\left\{\sqrt{\frac{2}{3}}\omega_0 t+\psi\right\}-\sqrt{3}\sqrt{2b^2\omega_0^2+3(b^2+\alpha)^2}}{\omega_0\sqrt{\omega_0^2+6\alpha}}\right\}.$$

If the cross section of the ellipsoid is noncircular (or the same applies to the cylinder), it is possible to integrate the relationships corresponding to the conservation of momentum and vorticity. As above, we assume that the F_{xx}, F_{yy}, F_{xy}, F_{yx} elements are different from zero. Equations (4) are then written as

$$\dot{F}_{xx}F_{xy}+\dot{F}_{yx}F_{yy}-F_{xx}\dot{F}_{xy}-F_{yx}\dot{F}_{yy}=-\left(\frac{eB}{mc}\right)(F_{xx}F_{yy}-F_{xy}F_{yx})+C_2,$$
$$F_{xx}\dot{F}_{yx}+F_{xy}\dot{F}_{yy}-F_{yy}\dot{F}_{xy}-F_{yx}\dot{F}_{xx}=-\frac{1}{2}\left(\frac{eB}{mc}\right)(F_{xx}^2+F_{yy}^2+F_{xy}^2+F_{yx}^2)+C_4.$$

We successively add and subtract these relations and perform a transformation to get

$$\left(\frac{F_{xy}-F_{yx}}{F_{xx}+F_{yy}}\right)^{\cdot}=\frac{eB}{2mc}\left[1+\left(\frac{F_{xy}-F_{yx}}{F_{xx}+F_{yy}}\right)^2\right]+\frac{C_2+C_4}{(F_{xx}+F_{yy})^2},$$
$$\left(\frac{F_{xx}-F_{yy}}{F_{xy}+F_{yx}}\right)^{\cdot}=\frac{eB}{2mc}\left[1+\left(\frac{F_{xx}-F_{yy}}{F_{xy}+F_{yx}}\right)^2\right]+\frac{C_2-C_4}{(F_{xy}+F_{yx})^2}, \tag{12}$$

which are readily integrated with appropriate initial conditions. In fact, it is possible to choose a set of initial conditions such that $C_2 = C_4 = 0$, and integration of (12) than gives

$$\frac{F_{xy}-F_{yx}}{F_{xx}+F_{yy}}=\tan\left(\frac{\omega_0 t}{2}+\psi_1\right),\quad \frac{F_{xx}-F_{yy}}{F_{xy}+F_{yx}}=\tan\left(\frac{\omega_0 t}{2}+\psi_2\right). \tag{13}$$

Equations (13) allow us to reduce the number of unknowns to a minimum. In particular, for $\psi_1 - \psi_2 = \pi/2$ we introduce $F_{xx} = R\sin\chi$, $F_{yy} = R\cos\chi$, $\delta = \omega_0 t/2 + \psi_1$ to get $F_{xy} = F_{xx}\tan\delta$, $F_{yx} = -F_{yy}\tan\delta$, $R = \sqrt{u\cos^2\delta}$ and

$$\dot{\chi}^2=\frac{1}{u\cos^2\delta}\left\{2C_1\cos^2\delta-(R^{\cdot})^2-\frac{\omega_0}{2}\tan\delta\,(R^2)^{\cdot}-\frac{\omega_0^2 u}{4}-\dot{F}_{zz}^2\cos^2\delta-\frac{4\alpha\cos^2\delta}{\gamma-1}\left[\frac{\cos^2\delta}{F_{zz}R^2\sin\chi\cos\chi}\right]^{\gamma-1}\right\}. \tag{14}$$

Equations (14) are very convenient for numerical analysis and correspond to the motion of a triaxial ellipsoid rotating in a magnetic field coincident with one of the axes.

The above particular cases correspond to various relationships between magnetic field (characteristic value ω_0) and the gas pressure (characteristic value α). The case of (8) corresponds to a fairly strong magnetic field, which is capable of retaining a rotating gas cloud. The case of (9) corresponds to a weak magnetic field that does not substantially influence the expansion of the cloud (after the compression part). It is possible to derive the general behavior, e.g., in the compression and escape of a rotating charged gas column (ellipsoid) in a longitudinal magnetic field by applying solutions of the type of (8) or (9) to the corresponding areas. The equations also allow one to construct more complex exact solutions corresponding to cases encountered in practice, in particular variations in the form of F_{zz}.

The applications in astrophysics and plasma physics are accompanied by independent interest in the results of integrating (2), since there are cases that are fairly complicated but which enable one to obtain solutions in elementary functions or which allow one to refer the final integrals to tabulated ones.

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